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LETTER TO THE EDITOR

Recursion operator for the stationary Nizhnik–Veselov–Novikov equation

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Abstract
We present a new general construction of a recursion operator from the zero-curvature representation. Using it, we find a recursion operator for the stationary Nizhnik–Veselov–Novikov equation and present a few low-order symmetries generated with the help of this operator.

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In this letter, we suggest a new method for the construction of a recursion operator using the zero-curvature representation. Unlike the majority of the hitherto known methods (see e.g. [1–4] and references therein), ours gives not only the recursion operator, but also its inverse, leading to the 'negative' part of the hierarchy of the system in question. What is more, it is immediately applicable to both evolutionary and non-evolutionary systems. We apply the method to the stationary Nizhnik–Veselov–Novikov (NVN) equation for which no recursion operator has been found so far.

Let $F = 0$ be a system of PDEs in two independent variables x, y for the unknown n-component vector function $\boldsymbol{u} = (u^1, \ldots, u^n)^T$, where the superscript 'T' denotes matrix transposition. Let this system have a zero-curvature representation $D_y(A) - D_x(B) + [A, B] = 0$, where A and B take values in a (matrix) Lie algebra $\mathfrak{g}$ and depend on λ, x, y, and $\boldsymbol{u}$ and its derivatives. Here D_x, D_y are the operators of total x- and y-derivatives (see e.g. ch 2 of [5], and [6]). Note that A and B may involve an essential (spectral) parameter λ.

Consider a function P (possibly vector- or matrix-valued) of $x, y, \boldsymbol{u}$ and its derivatives. Then the directional derivative of P along an n-component vector $\boldsymbol{U} = (U^1, \ldots, U^n)^T$ is given by $P'[\boldsymbol{U}] = \sum_{\alpha=1}^{n} \sum_{i,j=0}^{\infty} \left(\partial P \big/ \partial u_{ij}^{\alpha}\right) D_x^i D_y^j (U^\alpha)$, where $u_{00}^{\alpha} \equiv u^\alpha$, $u_{ij}^{\alpha} = \partial^{i+j} u^\alpha / \partial x^i \partial y^j$ (cf e.g. [7]). In [6], $P'[\boldsymbol{U}]$ is called a linearization and denoted by $\ell_P \boldsymbol{U}$.

Let $\boldsymbol{U}$ be a symmetry of the system $F = 0$, that is, let $\boldsymbol{U}$ satisfy $F'[\boldsymbol{U}] = 0$ on the solution manifold of $F = 0$ [5, 6]. Consider a $\mathfrak{g}$-valued solution S of the system

$$D_x(S) - [A, S] = \tilde{A} \equiv A'[\boldsymbol{U}] \qquad D_y(S) - [B, S] = \tilde{B} \equiv B'[\boldsymbol{U}]. \tag{1}$$

Assume that we have found n linear combinations $\tilde{U}^\alpha = \sum_{i,j} a^{\alpha,ij} S_{ij}$ of entries S_{ij} of S, $\alpha = 1, \ldots, n$, with the property that $\tilde{\boldsymbol{U}} = (\tilde{U}^1, \ldots, \tilde{U}^n)^T$ is another symmetry of $F = 0$. Then the linear operator $\mathfrak{R}_0$ defined by $\tilde{\boldsymbol{U}} = \mathfrak{R}_0(\boldsymbol{U})$ is a recursion operator for the system $F = 0$ in Guthrie's [8] sense. The coefficients $a^{\alpha,ij}$ may depend on λ, x, y and $\boldsymbol{u}$ and its derivatives.

However, testing the above scheme on a number of known examples such as the KdV or the Harry Dym equation shows that $\mathfrak{R}_0$ generates the nonlocal symmetries that belong to the 'negative' part of the hierarchy of $F = 0$. Then we should, if possible, invert $\mathfrak{R}_0$ in order to obtain a 'conventional' recursion operator $\mathfrak{R}$ which will generate the 'positive' local part of the hierarchy in question. The inversion is an algorithmic process described in [8]. Note [9] that if the coefficients of the recursion operator are local, then so are the coefficients of its inverse.

Let us now apply this procedure to the stationary NVN equation

$$u_{yyy} = u_{xxx} - 3(vu)_x + 3(wu)_y \qquad w_x = u_y \qquad v_y = u_x \tag{2}$$

recently studied by Ferapontov [10] (see also Rogers and Schief [11]) in connection with isothermally asymptotic surfaces in projective differential geometry.

The stationary NVN equation is a reduction of the NVN equation [12, 13]

$$u_t = u_{xxx} - u_{yyy} - 3(vu)_x + 3(wu)_y \qquad w_x = u_y \qquad v_y = u_x \tag{3}$$

obtained upon assuming that u, v, w are independent of t. The latter is well known to be integrable via the inverse scattering transform, as it has the Lax pair

$$\psi_{xy} = u\psi \qquad \psi_t = \psi_{xxx} - \psi_{yyy} - 3v\psi_x + 3w\psi_y. \tag{4}$$

The NVN equation (3) is the first member of the hierarchy describing the deformations preserving the zero-energy level of the two-dimensional Schrödinger operator [13]. It also naturally arises in the theory of surfaces (see [11] and references therein) and its modified version appears in string theory [14, 15].

Upon setting [11] $\psi = \tilde{\psi}\exp(\lambda t)$, where λ is a constant, the Lax pair (4) can be transformed into a zero-curvature representation for (2) of the form $D_y(A) - D_x(B) + [A, B] = 0$. This representation involves an essential parameter λ, and the matrices A and B belong to the semisimple Lie algebra sl_6 of traceless 6×6 matrices. They read

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ u & 0 & 0 & 0 & 0 & 0 \\ A_{41} & \lambda & u_y & 0 & 0 & -u \\ 0 & 3v & 0 & -1 & 0 & 0 \\ u_y & 0 & u & 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ u & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ B_{41} & u_x & 0 & 0 & -u & 0 \\ u_x & u & 0 & 0 & 0 & 0 \\ \lambda & 0 & 3w & -1 & 0 & 0 \end{pmatrix} \tag{5}$$

where $A_{41} = -u_{yy} + 3wu$, $B_{41} = -u_{xx} + 3vu$.

Let $\boldsymbol{U} = (U, V, W)^T$ be a symmetry of (2), i.e. let U, V, W satisfy

$$\begin{aligned} &D_y^3 U = D_x^3 U + 3[wD_yU + u(D_yW - D_xV) - u_xV + u_yW + (w_y - v_x)U - vD_xU] \\ &D_xW = D_yU \qquad D_yV = D_xU. \end{aligned} \tag{6}$$

Consider a traceless 6×6 matrix S that solves (1), where A, B are given by (5), and

$$\tilde{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ u & 0 & 0 & 0 & 0 & 0 \\ \tilde{A}_{41} & 0 & D_yU & 0 & 0 & -U \\ 0 & 3V & 0 & 0 & 0 & 0 \\ D_yU & 0 & U & 0 & 0 & 0 \end{pmatrix} \qquad \tilde{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ U & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \tilde{B}_{41} & D_xU & 0 & 0 & -U & 0 \\ D_xU & U & 0 & 0 & 0 & 0 \\ 0 & 0 & 3W & 0 & 0 & 0 \end{pmatrix}$$

are the directional derivatives of the matrices A and B (5) along the vector $(U, V, W)^T$, $\tilde{A}_{41} = -D_y^2(U) + 3wU + 3uW$, $\tilde{B}_{41} = -D_x^2(U) + 3vU + 3uV$.

The next step is to find linear combinations $\tilde{U}$, $\tilde{V}$, $\tilde{W}$ of entries of S that solve (6). A straightforward but tiresome computation shows that for A and B (5) these are $\tilde{U} = -S_{35} + S_{26}$, $\tilde{V} = S_{54} + S_{12}$, $\tilde{W} = -S_{13} - S_{64}$, i.e. if $\boldsymbol{U} = (U, V, W)^T$ is a symmetry of (2), then so is $\tilde{\boldsymbol{U}} = (\tilde{U}, \tilde{V}, \tilde{W})^T$. Hence, the linear operator $\mathfrak{R}_0$ mapping $\boldsymbol{U}$ to $\tilde{\boldsymbol{U}}$ is a recursion operator for (2).

However, the application of $\mathfrak{R}_0$ to the simplest symmetries of (2), e.g., to the zero symmetry, yields nonlocal symmetries of (2), so we should invert $\mathfrak{R}_0$ in order to obtain a recursion operator $\tilde{\mathfrak{R}} = \mathfrak{R}_0^{-1}$ generating hierarchies of local symmetries for (2).

It turns out that our $\mathfrak{R}_0$ is invertible only for $\lambda \neq 0$. Inverting $\mathfrak{R}_0$ involves solving a system of algebraic and differential equations for the components of $\tilde{\mathfrak{R}}$, which is a fairly tiresome but algorithmic process. For the sake of simplicity, we set all the integration constants to zero. Then $\mathfrak{R} = \lambda\tilde{\mathfrak{R}} - \frac{1}{2}\lambda^2\mathrm{id}$, where id is the identity operator, is independent of λ and provides a conventional recursion operator for (2).

The action of $\mathfrak{R}$ on a symmetry $\boldsymbol{U} = (U, V, W)^T$ of (2) is given by $\mathfrak{R}(\boldsymbol{U}) = \mathfrak{L}(\boldsymbol{U}) + \mathfrak{M}\vec{Z}$. Here $\vec{Z} = (Z_1, Z_2, Z_3, Z_4, Z_5)^T$ is a general solution of the system

$$
\begin{aligned}
&D_x Z_1 = U \qquad D_y Z_1 = W \qquad D_x Z_2 = V \qquad D_y Z_2 = U\\
&D_x Z_3 = D_y^2 U - 3(Wu + Uw) \qquad D_y Z_3 = D_x^2 U - 3(Vu + Uv)\\
&D_x Z_4 = -\tfrac{1}{3} v D_y^2 U + \tfrac{1}{3} u_x D_y U + \left(-\tfrac{1}{3} u_{xy} + u^2 + vw\right) U + \left(-\tfrac{1}{3} u_{yy} + \tfrac{1}{3} v_{xx} + 2uw - v^2\right) V\\
&\qquad + uvW + uu_x Z_1 + (wu_x + uu_y - vv_x) Z_2\\
&D_y Z_4 = -\tfrac{1}{3} v D_x^2 U + \tfrac{1}{3} v_x D_x U + \left(-\tfrac{1}{3} u_{yy} + 2uw\right) U + uvV + uu_y Z_1 + u^2 W\\
&\qquad + (-vu_x + wu_y + uw_y) Z_2\\
&D_x Z_5 = -\tfrac{1}{3} w D_y^2 U + \tfrac{1}{3} w_y D_y U + \left(-\tfrac{1}{3} u_{xx} + 2uv\right) U + u^2 V + uwW\\
&\qquad + (vu_x - wu_y + uv_x) Z_1 + uu_x Z_2\\
&D_y Z_5 = -\tfrac{1}{3} w D_x^2 U + \tfrac{1}{3} u_y D_x U + \left(-\tfrac{1}{3} u_{xy} + u^2 + vw\right) U + uwV\\
&\qquad + \left(-\tfrac{1}{3} u_{xx} + \tfrac{1}{3} w_{yy} + 2uv - w^2\right) W + (uu_x + vu_y - ww_y) Z_1 + uu_y Z_2.
\end{aligned}
\tag{7}
$$

Note that this system is compatible if and only if $\boldsymbol{U}$ solves (6).

The operators $\mathfrak{L}$ and $\mathfrak{M}$ are of the form

$$
\mathfrak{L} = \begin{pmatrix} \mathfrak{L}_{11} & \mathfrak{L}_{12} & \mathfrak{L}_{13}\\ \mathfrak{L}_{21} & \mathfrak{L}_{22} & \mathfrak{L}_{23}\\ \mathfrak{L}_{31} & \mathfrak{L}_{32} & \mathfrak{L}_{33} \end{pmatrix} \qquad \mathfrak{M} = \begin{pmatrix} \mathfrak{M}_{11} & \mathfrak{M}_{12} & \mathfrak{M}_{13} & \mathfrak{M}_{14} & \mathfrak{M}_{15}\\ \mathfrak{M}_{21} & \mathfrak{M}_{22} & \mathfrak{M}_{23} & \mathfrak{M}_{24} & \mathfrak{M}_{25}\\ \mathfrak{M}_{31} & \mathfrak{M}_{32} & \mathfrak{M}_{33} & \mathfrak{M}_{34} & \mathfrak{M}_{35} \end{pmatrix}
$$

$$
\begin{aligned}
\mathfrak{L}_{11} = {}& D_x^6 - 6vD_x^4 - \tfrac{25}{9} u D_x^2 D_y^2 - 15 v_x D_x^3 - \tfrac{2}{9} u_y D_x^2 D_y - \tfrac{29}{9} u_x D_x D_y^2 + \left(-\tfrac{5}{3} u_{yy} - 18 v_{xx}\right.\\
&\left. + \tfrac{40}{3} uw + 9v^2\right) D_x^2 + 9u^2 D_x D_y + \left(-\tfrac{5}{3} u_{xx} + \tfrac{13}{3} uv\right) D_y^2 + \left(-3u_{xyy} - 12 v_{xxx}\right.\\
&\left. + \tfrac{56}{3} wu_x + 26uu_y + 27vv_x\right) D_x + \left(26uu_x + \tfrac{2}{3} vu_y\right) D_y - 3u_{xxyy} - 3v_{xxxx}\\
&+ 14wu_{xx} + 20uu_{xy} + 5vu_{yy} + 9vv_{xx} + \tfrac{77}{3} u_x u_y + 9v_x^2 - 4u^3 - 28uvw\\
\mathfrak{L}_{12} = {}& -\tfrac{28}{9} u D_x^4 - \tfrac{106}{9} u_x D_x^3 + \left(-\tfrac{55}{3} u_{xx} + \tfrac{32}{3} uv\right) D_x^2 + \left(-\tfrac{44}{3} u_{xxx} + \tfrac{74}{3} vu_x + 18 uv_x\right) D_x\\
&- 6u_{xxxx} + 19vu_{xx} + 4uu_{yy} + 10uv_{xx} + \tfrac{79}{3} u_x v_x + \tfrac{2}{3} u_y^2 - 12u^2 w - 4uv^2\\
\mathfrak{L}_{13} = {}& -\tfrac{1}{9} u D_y^4 + \tfrac{2}{9} u_y D_y^3 + \left(-\tfrac{1}{3} u_{yy} + \tfrac{5}{3} uw\right) D_y^2 + \left(\tfrac{1}{3} u_{xxx} - vu_x - \tfrac{4}{3} wu_y - uv_x + uw_y\right) D_y\\
&+ 13uu_{xx} + wu_{yy} + uw_{yy} + \tfrac{29}{3} u_x^2 - \tfrac{2}{3} u_y w_y - 12u^2 v - 4uw^2
\end{aligned}
$$

$$\begin{aligned}
\mathfrak{L}_{21} = {}& \tfrac{28}{27}D_x^4D_y^2 - \tfrac{28}{9}wD_x^4 - 6uD_x^3D_y - \tfrac{32}{9}vD_x^2D_y^2 - \tfrac{86}{9}u_yD_x^3 - \tfrac{134}{9}u_xD_x^2D_y - \tfrac{58}{9}v_xD_xD_y^2 \\
&+ \left(-\tfrac{148}{9}u_{xy} + \tfrac{28}{3}u^2 + \tfrac{32}{3}vw\right)D_x^2 + \left(-\tfrac{154}{9}u_{xx} + \tfrac{46}{3}uv\right)D_xD_y + \left(-\tfrac{16}{9}u_{yy}\right. \\
&\left. - \tfrac{28}{9}v_{xx} + \tfrac{16}{3}uw + \tfrac{4}{3}v^2\right)D_y^2 + \left(-14u_{xxy} + \tfrac{89}{3}uu_x + \tfrac{50}{3}vu_y + \tfrac{58}{3}wv_x\right)D_x \\
&+ \left(-\tfrac{86}{9}u_{xxx} + \tfrac{58}{3}vu_x + \tfrac{59}{3}uv_x\right)D_y - 6u_{xxxy} + 20uu_{xx} + \tfrac{40}{3}vu_{xy} + \tfrac{16}{3}wu_{yy} \\
&+ \tfrac{28}{3}wv_{xx} + \tfrac{47}{3}u_x^2 + 19u_yv_x - 12u^2v - 16uw^2 - 4v^2w \\
\mathfrak{L}_{22} = {}& -\tfrac{1}{27}D_x^6 + \tfrac{2}{3}vD_x^4 + v_xD_x^3 + \left(-\tfrac{28}{9}u_{yy} + \tfrac{11}{9}v_{xx} + \tfrac{28}{3}uw - 3v^2\right)D_x^2 + \left(-\tfrac{56}{9}u_{xyy}\right. \\
&\left. + \tfrac{10}{9}v_{xxx} + \tfrac{56}{3}wu_x + \tfrac{56}{3}uu_y - 7vv_x\right)D_x - \tfrac{40}{9}u_{xxyy} + \tfrac{5}{9}v_{xxxx} + \tfrac{40}{3}wu_{xx} \\
&+ \tfrac{52}{3}uu_{xy} + \tfrac{16}{3}vu_{yy} - \tfrac{16}{3}vv_{xx} + \tfrac{70}{3}u_xu_y - 2v_x^2 - 4u^3 - 16uvw + 4v^3 \\
\mathfrak{L}_{23} = {}& -\tfrac{1}{9}u_xD_y^3 + \left(\tfrac{1}{3}u_{xy} - \tfrac{1}{3}u^2\right)D_y^2 + \left(-\tfrac{2}{3}u_{xyy} + \tfrac{5}{3}wu_x + 2uu_y\right)D_y - \tfrac{19}{9}u_{xxxx} + \tfrac{23}{3}vu_{xx} \\
&- wu_{xy} + \tfrac{13}{3}uu_{yy} + \tfrac{19}{3}uv_{xx} + \tfrac{40}{3}u_xv_x + \tfrac{4}{3}u_xw_y - 12u^2w - 4uv^2 \\
\mathfrak{L}_{31} = {}& \tfrac{28}{27}D_x^5D_y - 6uD_x^4 - \tfrac{56}{9}vD_x^3D_y - \tfrac{4}{9}wD_x^2D_y^2 - \tfrac{142}{9}u_xD_x^3 + \left(-\tfrac{28}{3}v_x - \tfrac{2}{9}w_y\right)D_x^2D_y \\
&- \tfrac{22}{9}u_yD_xD_y^2 + \left(-\tfrac{184}{9}u_{xx} + \tfrac{70}{3}uv + \tfrac{4}{3}w^2\right)D_x^2 + \left(-\tfrac{14}{9}u_{yy} - \tfrac{28}{3}v_{xx} + \tfrac{20}{3}uw\right. \\
&\left. + \tfrac{28}{3}v^2\right)D_xD_y + \left(-\tfrac{8}{9}u_{xy} + \tfrac{2}{3}u^2 + \tfrac{4}{3}vw\right)D_y^2 + \left(-\tfrac{142}{9}u_{xxx} + \tfrac{116}{3}vu_x + \tfrac{28}{3}wu_y\right. \\
&\left. + \tfrac{82}{3}uv_x + uw_y\right)D_x + \left(-\tfrac{14}{9}u_{xyy} - \tfrac{28}{9}v_{xxx} + \tfrac{20}{3}wu_x + \tfrac{37}{3}uu_y + \tfrac{28}{3}vv_x\right. \\
&\left. + \tfrac{2}{3}vw_y\right)D_y - 6u_{xxxx} + \tfrac{70}{3}vu_{xx} + \tfrac{14}{3}wu_{xy} + \tfrac{8}{3}uu_{yy} + \tfrac{52}{3}uv_{xx} + \tfrac{82}{3}u_xv_x \\
&+ \tfrac{1}{3}u_xw_y + 7u_y^2 - 12u^2w - 16uv^2 - 4vw^2 \\
\mathfrak{L}_{32} = {}& -\tfrac{29}{9}u_yD_x^3 + \left(-9u_{xy} + \tfrac{25}{3}u^2\right)D_x^2 + \left(-10u_{xxy} + \tfrac{88}{3}uu_x + 11vu_y\right)D_x - \tfrac{47}{9}u_{xxxy} \\
&+ \tfrac{58}{3}uu_{xx} + \tfrac{44}{3}vu_{xy} + \tfrac{4}{3}wu_{yy} + \tfrac{49}{3}u_x^2 + \tfrac{23}{3}u_yv_x + \tfrac{2}{3}u_yw_y - 12u^2v - 4uw^2 \\
\mathfrak{L}_{33} = {}& -\tfrac{1}{27}D_y^6 + \tfrac{2}{3}wD_y^4 + w_yD_y^3 + \left(\tfrac{11}{9}w_{yy} - 3w^2\right)D_y^2 + \left(\tfrac{10}{9}w_{yyy} - 7ww_y\right)D_y - \tfrac{4}{3}u_{xxyy} \\
&+ \tfrac{5}{9}w_{yyyy} + \tfrac{16}{3}wu_{xx} + \tfrac{26}{3}uu_{xy} + 4vu_{yy} - \tfrac{16}{3}ww_{yy} + \tfrac{40}{3}u_xu_y - 2w_y^2 \\
&- 4u^3 - 16uvw + 4w^3 \\
\mathfrak{M}_{11} = {}& -u_{xxxyy} + 2wu_{xxx} + \tfrac{14}{3}uu_{xxy} + 3vu_{xyy} + \tfrac{1}{3}uw_{yyy} + \tfrac{23}{3}u_yu_{xx} + 9u_xu_{xy} + 3v_xu_{yy} \\
&+ w_yu_{yy} - \tfrac{2}{3}u_yw_{yy} - 4u^2u_x - 6vwu_x - 8uvu_y - 6uwv_x - 4uww_y \\
\mathfrak{M}_{12} = {}& -u_{xxxxx} + 5vu_{xxx} + \tfrac{5}{3}uu_{xyy} + \tfrac{10}{3}uv_{xxx} + 10v_xu_{xx} + \tfrac{5}{3}u_xu_{yy} + \tfrac{25}{3}u_xv_{xx} - 8uwu_x \\
&- 6v^2u_x - 4u^2u_y - 10uvv_x \\
\mathfrak{M}_{13} = {}& -\tfrac{2}{3}u_{xxx} + 2vu_x + 2uv_x \qquad \mathfrak{M}_{14} = -2u_x \qquad \mathfrak{M}_{15} = -2u_y \\
\mathfrak{M}_{21} = {}& -u_{xxxxy} + 5uu_{xxx} + 3vu_{xxy} + 2wu_{xyy} + \tfrac{32}{3}u_xu_{xx} + 6v_xu_{xy} - w_yu_{xy} + u_yu_{yy} \\
&+ 3u_yv_{xx} + \tfrac{1}{3}u_xw_{yy} - 9uvu_x - 6w^2u_x - 9uwu_y - 5u^2v_x + u^2w_y \\
\mathfrak{M}_{22} = {}& -\tfrac{10}{9}u_{xxxyy} + \tfrac{1}{9}v_{xxxxx} + \tfrac{10}{3}wu_{xxx} + 5uu_{xxy} + \tfrac{10}{3}vu_{xyy} - \tfrac{5}{3}vv_{xxx} + \tfrac{25}{3}u_yu_{xx} + 10u_xu_{xy} \\
&+ \tfrac{10}{3}v_xu_{yy} - \tfrac{5}{3}v_xv_{xx} - 10vwu_x - 4u^2u_x - 10uvu_y + 4v^2v_x - 8uwv_x \\
\mathfrak{M}_{23} = {}& -\tfrac{2}{3}u_{xyy} + 2wu_x + 2uu_y \qquad \mathfrak{M}_{24} = -2v_x \qquad \mathfrak{M}_{25} = -2u_x \\
\mathfrak{M}_{31} = {}& -\tfrac{10}{9}u_{xxxxx} + \tfrac{1}{9}w_{yyyyy} + \tfrac{20}{3}vu_{xxx} + \tfrac{5}{3}uu_{xyy} + \tfrac{10}{3}uv_{xxx} - \tfrac{5}{3}ww_{yyy} + 10v_xu_{xx} + \tfrac{5}{3}u_xu_{yy} \\
&+ 10u_xv_{xx} - \tfrac{5}{3}w_yw_{yy} - 10v^2u_x - 10uwu_x - 4u^2u_y - 10uvv_x + 4w^2w_y + 2uvw_y \\
\mathfrak{M}_{32} = {}& -u_{xxxxy} + 5uu_{xxx} + 5vu_{xxy} + 10u_xu_{xx} + 5v_xu_{xy} + \tfrac{5}{3}u_yu_{yy} + \tfrac{10}{3}u_yv_{xx} \\
&- 15uvu_x - 6v^2u_y - 3uwu_y - 5u^2v_x + u^2w_y \\
\mathfrak{M}_{33} = {}& -\tfrac{2}{3}u_{xxy} + 2uu_x + 2vu_y \qquad \mathfrak{M}_{34} = -2u_y \qquad \mathfrak{M}_{35} = -2w_y.
\end{aligned}$$

Note that the above formula $\mathfrak{R}(\boldsymbol{U}) = \mathfrak{L}(\boldsymbol{U}) + \mathfrak{M}\vec{Z}$ defines a recursion operator in the sense of Guthrie [8], and the system (7) defines a covering [6] over (6). Formally, we could express Z_i from (7) as $Z_1 = D_x^{-1}U$, $Z_2 = D_x^{-1}V$, etc, and thus write $\mathfrak{R}$ as an integro-differential operator as has become a tradition in the literature (see e.g. [3, 5, 7]). However, if we drop the y-part of (7), we encounter certain difficulties in constructing new symmetries (cf e.g. [8, 16, 17]).

As integrating (7) involves arbitrary constants, we have $\mathfrak{R}(0) = c_1\boldsymbol{S}_1 + c_2\boldsymbol{S}_2 + \cdots + c_5\boldsymbol{S}_5$, where c_i are constants, and $\boldsymbol{S}_1, \ldots, \boldsymbol{S}_5$ are symmetries of (2) of the following form:

$$\boldsymbol{S}_1 = \begin{pmatrix} \mathfrak{M}_{11} + \frac{1}{2}u^2\mathfrak{M}_{14} + \left(uv - \frac{1}{2}w^2\right)\mathfrak{M}_{15} \\ \mathfrak{M}_{21} + \frac{1}{2}u^2\mathfrak{M}_{24} + \left(uv - \frac{1}{2}w^2\right)\mathfrak{M}_{25} \\ \mathfrak{M}_{31} + \frac{1}{2}u^2\mathfrak{M}_{34} + \left(uv - \frac{1}{2}w^2\right)\mathfrak{M}_{35} \end{pmatrix}$$

$$\boldsymbol{S}_2 = \begin{pmatrix} \mathfrak{M}_{12} + \left(uw - \frac{1}{2}v^2\right)\mathfrak{M}_{14} + \frac{1}{2}u^2\mathfrak{M}_{15} \\ \mathfrak{M}_{22} + \left(uw - \frac{1}{2}v^2\right)\mathfrak{M}_{24} + \frac{1}{2}u^2\mathfrak{M}_{25} \\ \mathfrak{M}_{32} + \left(uw - \frac{1}{2}v^2\right)\mathfrak{M}_{34} + \frac{1}{2}u^2\mathfrak{M}_{35} \end{pmatrix}$$

$$\boldsymbol{S}_3 = \begin{pmatrix} -u_{xxx} + 3(vu_x + uv_x) \\ -u_{xyy} + 3(wu_x + uu_y) \\ -u_{xxy} + 3(vu_y + uu_x) \end{pmatrix}$$

$$\boldsymbol{S}_4 = \boldsymbol{u}_x \equiv \begin{pmatrix} u_x \\ v_x \\ w_x \end{pmatrix} \qquad \boldsymbol{S}_5 = \boldsymbol{u}_y \equiv \begin{pmatrix} u_y \\ v_y \\ w_y \end{pmatrix}.$$

The repeated application of $\mathfrak{R}$ to $\boldsymbol{S}_1, \ldots, \boldsymbol{S}_5$ produces five hierarchies of symmetries of the stationary NVN equation (2), which can be visualized as follows (numbers in the top line denote the orders of symmetries):

$$\begin{array}{ccccccccc}
 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & \\
 & \boldsymbol{S}_5 & \longrightarrow & \longrightarrow & \mathfrak{R}\boldsymbol{S}_5 & \longrightarrow & \longrightarrow & \mathfrak{R}^2\boldsymbol{S}_5 & \cdots \\
 & \boldsymbol{S}_4 & \longrightarrow & \longrightarrow & \mathfrak{R}\boldsymbol{S}_4 & \longrightarrow & \longrightarrow & \mathfrak{R}^2\boldsymbol{S}_4 & \cdots \\
0 & \longrightarrow & \boldsymbol{S}_3 & \longrightarrow & \longrightarrow & \mathfrak{R}\boldsymbol{S}_3 & \longrightarrow & \longrightarrow & \cdots \\
 & \longrightarrow & \longrightarrow & \boldsymbol{S}_2 & \longrightarrow & \longrightarrow & \mathfrak{R}\boldsymbol{S}_2 & \longrightarrow & \cdots \\
 & \longrightarrow & \longrightarrow & \boldsymbol{S}_1 & \longrightarrow & \longrightarrow & \mathfrak{R}\boldsymbol{S}_1 & \longrightarrow & \cdots
\end{array}$$

We conjecture that all these symmetries are local and commute, as is the case for the symmetries of orders $1, 3, 5, \ldots, 11$.

Note that (2) has a scaling symmetry $\boldsymbol{S} = x\boldsymbol{u}_x + y\boldsymbol{u}_y + 2\boldsymbol{u}$. The application of $\mathfrak{R}$ to $\boldsymbol{S}$ yields a nonlocal symmetry of seventh order, which we conjecture to be a master symmetry for (2), cf [18], meaning that commuting $\mathfrak{R}(\boldsymbol{S})$ with any symmetry belonging to one of the five hierarchies, described above, yields (up to a constant multiplier) the next member of the same hierarchy. The repeated application of $\mathfrak{R}$ to $\boldsymbol{S}$ yields an infinite hierarchy of nonlocal symmetries for (2).

We believe that $\mathfrak{R}$ is hereditary in the sense of [19], but we have not yet checked this because of the huge amount of computations involved.

As a final remark, let us mention the nonstandard structure of nonlocal terms of $\mathfrak{R}$ in (7): they involve the derivatives of components of the symmetry, which is quite unusual (cf e.g. [20] for another example of this kind and [21] for a comprehensive list of integrable systems in (1 + 1) dimensions and their recursion operators known today).

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References

[1] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 *Stud. Appl. Math.* **53** 249–315

[2] Fokas A S and Anderson R L 1982 *J. Math. Phys.* **23** 1066–73

[3] Konopelchenko B G 1987 *Nonlinear Integrable Equations. Recursion Operators, Group-Theoretical and Hamiltonian Structures of Soliton Equations* (Berlin: Springer)
Konopelchenko B G 1988 *Inverse Probl.* **4** 785–814

[4] Gürses M, Karasu A and Sokolov V V 1999 *J. Math. Phys.* **40** 6473–90

[5] Olver P J 1993 *Applications of Lie Groups to Differential Equations* (New York: Springer)

[6] Bocharov A V *et al* 1999 *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics* (Providence, RI: American Mathematical Society)

[7] Błaszak M 1998 *Multi-Hamiltonian Theory of Dynamical Systems* (Heidelberg: Springer)

[8] Guthrie G A 1994 *Proc. R. Soc.* A **446** 107–14

[9] Marvan M 1996 *Differential Geometry and Applications (Brno, 1995)* ed J Janyška, I Kolář and J Slovák (Brno: Masaryk University) pp 393–402 (available at http://www.emis.de/proceedings)

[10] Ferapontov E V 1999 *Differ. Geom. Appl.* **11** 117–28

[11] Rogers C and Schief W K 2002 *Bäcklund and Darboux Transformations. Geometry and Modern Applications in Soliton Theory* (Cambridge: Cambridge University Press)

[12] Nizhnik L P 1980 *Sov. Math. Dokl.* **25** 706–8

[13] Veselov A P and Novikov S P 1984 *Sov. Math. Dokl.* **30** 588–91

[14] Konopelchenko B G 1997 *Phys. Lett.* B **414** 58–64

[15] Yamagishi K 1999 *Phys. Lett.* B **454** 31–7

[16] Sanders J A and Wang J P 2001 *Physica* D **149** 1–10

[17] Sergyeyev A 2000 *Proc. Sem. Differ. Geom.* ed D Krupka (Opava: Silesian University in Opava) pp 159–73 (*Preprint* nlin.SI/0012011)

[18] Fokas A S and Fuchssteiner B 1981 *Phys. Lett.* A **86** 341–5
Fuchssteiner B 1983 *Prog. Theor. Phys.* **70** 1508–22

[19] Fuchssteiner B and Fokas A S 1981 *Physica* D **4** 47–66

[20] Karasu (Kalkanli) A, Karasu A and Sakovich S Yu 2002 A strange recursion operator for a new integrable system of coupled Korteweg–de Vries equations *Preprint* nlin.SI/0203036

[21] Wang J P 2002 *J. Nonlinear Math. Phys.* **9** (suppl 1) 213–33